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The Effective Lagrangian of QED with a Magnetic Charge

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Abstract

The effective Lagrangian of QED coupled to dyons is calculated. The resulting generalization of the Euler-Heisenberg Lagrangian contains non-linear P- and T-nonivariant (but C invariant) terms corresponding to the virtual pair creation of dyons. As examples, the amplitudes for photon splitting and photon coalescence are calculated.

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The calculation of quantum corrections due to the virtual pair creation of dyons is a very difficult problem because the standard diagram technique is not valid in this case. The difficulty is connected both to the large value of the magnetic charge of the dyon and the lack of a consistent local Lagrangian formulation of electrodynamics with two types of charge [1]. So, there is no possibility to use a perturbation expansion in a coupling constant. But one can apply the loop expansion which is just an expansion in powers of the Planck constant $\hbar$.

Actually, it is known (see, e.g. [2]), that the one-loop quantum correction to the QED Lagrangian can be calculated without the use of perturbation methods. The correction is just the change in the vacuum energy in an external field. Let us review the simple case of weak constant parallel electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$. We impose the conditions $e|\boldsymbol{E}|/m^2 \ll 1$ and $e|\boldsymbol{H}|/m^2 \ll 1$ such that the creation of particles is not possible. In this case the one-loop correction can be calculated by summing the one-particle modes — the solutions of the Dirac equation in the external electromagnetic field — over all quantum numbers [2], [3]. For example, if there is just a magnetic field, $\boldsymbol{H} = (0, 0, H)$, the corresponding equation is

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi(x) = 0 \tag{1}$$

where the electromagnetic potential is $A^\mu = (0, -Hy, 0, 0)$. The solution of this equation gives the energy levels of an electron in a magnetic field [4], [5]

$$\varepsilon_n = \sqrt{m^2 + eH(2n - 1 + s) + k^2} \tag{2}$$

where $n = 0, 1, 2 \ldots$, $s = \pm 1$, and k is the electron momentum along the field. In this case the correction to the Lagrangian is [2], [4]

$$\begin{aligned} \Delta L_H &= \frac{eH}{2\pi^2} \int\limits_0^\infty \mathrm{d}k \left[(m^2 + k^2)^{1/2} + 2\sum_{n=1}^{\infty} (m^2 + 2eHn + k^2)^{1/2} \right] \\ &= -\frac{1}{8\pi^2} \int\limits_0^\infty \frac{\mathrm{d}s}{s^3} e^{-m^2 s} \left[(esH) \coth{(esH)} - 1 - \frac{1}{3} e^2 s^2 H^2 \right], \end{aligned} \tag{3}$$

where the terms independent of the external field $\boldsymbol{H}$ are dropped and a standard renormalization of the electron charge has been made [2].

It is known [5] that if we consider simultaneously magnetic ($\boldsymbol{H}$) and electric ($\boldsymbol{E}$) homogeneous fields, then equation (1), as well as its classical analogue can be separated into two uncoupled equations, each in two variables. Indeed, in this case we can take $A^\mu = (Ez, -Hy, 0, 0)$ and the interaction of an electron with the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ determined independently. For such a configuration of electromagnetic fields the correction to the Lagrangian is (see [2], p. 787)

$$\Delta L = \frac{eH}{2\pi^2} \sum_{n=1}^{\infty} \int\limits_0^\infty \mathrm{d}k\, \varepsilon_n^{(E)}(k). \tag{4}$$

Here $\varepsilon_n^{(E)}$ is the correction to the energy of an electron in the combined external magnetic and electric fields, which is in the first order proportional to e^2E^2.

So, the total Lagrangian is $L = L_0 + \Delta L$, where $L_0 = (\boldsymbol{E}^2 - \boldsymbol{H}^2)/2$ is just the Lagrangian of the free electromagnetic field in the tree approximation, and can be written as

$$L = \left(1 + \frac{\alpha}{3\pi}\int\limits_0^\infty \frac{\mathrm{d}s}{s} e^{-m^2 s}\right) \frac{\boldsymbol{E}^2 - \boldsymbol{H}^2}{2} + \Delta L'. \tag{5}$$

The logarithmic divergency can be removed by the standard renormalization of the external fields and the electron charge:

$$E_{\mathrm{reg}} = Z_3^{-1/2} E; \qquad H_{\mathrm{reg}} = Z_3^{-1/2} H; \qquad e_{\mathrm{reg}} = Z_3^{1/2} e, \tag{6}$$

where $Z_3^{-1} = 1 + \frac{\alpha}{3\pi}\int\limits_0^\infty \frac{ds}{s} e^{-m^2 s}$ is just the usual QED renormalization factor. Thus the finite part of the correction to the Lagrangian $\Delta L'$ can be written in terms of physical quantities as (see [2], p. 790)

$$\Delta L' = -\frac{1}{8\pi^2}\int\limits_0^\infty \frac{\mathrm{d}s}{s^3} e^{-m^2 s}\left[(esE)(esH)\ \cot(esE)\coth(esH) - 1\right], \tag{7}$$

which in the limit $E = 0$ reduces to the renormalized form of (3).

The series expansion of (7) in terms of the parameters $eE/m^2 \ll 1$, $eH/m^2 \ll 1$ yields the well known Euler-Heisenberg correction [6]:

$$\Delta L' \approx \frac{e^4}{360\pi^2 m^4}\Big[(\boldsymbol{H}^2 - \boldsymbol{E}^2)^2 + 7(\boldsymbol{H}\boldsymbol{E})^2\Big], \tag{8}$$

where $e^2 = \alpha$.[2]

Let us consider how the situation changes if we consider the virtual pair creation of dyons in the external electromagnetic field. Using an analogy with the classical Lorentz force on a dyon of velocity $\boldsymbol{v}$ with electric (Q) and magnetic (g) charges [1]

$$\boldsymbol{F} = Q\boldsymbol{E} + g\boldsymbol{H} + \boldsymbol{v} \times (Q\boldsymbol{H} - g\boldsymbol{E}), \tag{9}$$

we shall assume that the wave equation for this particle in an external electromagnetic field can be expressed as [8]

$$(i\gamma^\mu D_\mu - M)\psi(x) = 0, \tag{10}$$

[2]There is a misprint in ref. [2], they use $e^2 = \alpha$ for this formula, but elsewhere (p. 122) they have $e^2/4\pi = \alpha$.

where M is the dyon mass, and iD_μ a generalized momentum operator, with $D_\mu = \partial_\mu + iQA_\mu + igB_\mu$.[3]

The potential A_μ and its dual B_μ are defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \varepsilon_{\mu\nu\rho\sigma}\partial^\rho B^\sigma$ where $F_{\mu\nu}$ is the electromagnetic field strength tensor[4] and $\varepsilon_{0123} = 1$. The potentials in the case of constant parallel electric and magnetic fields can be expressed as

$$A^\mu = (Ez, -Hy, 0, 0), \qquad B^\mu = (Hz, Ey, 0, 0). \tag{11}$$

It is easily seen that the solution to the equation of motion for a dyon in an external electromagnetic field can be obtained from the solution to the equation for an electron (1) by the following substitution

$$eE \to QE + gH; \qquad eH \to QH - gE. \tag{12}$$

Using the same substitution as in Eqs. (5) and (7), we obtain the following expression for the quantum correction to the Lagrangian, due to the vacuum polarization caused by dyons:

$$L = \left(1 + \frac{Q^2}{12\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}e^{-M^2 s} - \frac{g^2}{12\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s}e^{-M^2 s}\right)\frac{\boldsymbol{E}^2 - \boldsymbol{H}^2}{2} + \Delta L', \tag{13}$$

where a total derivative has been dropped.

For the renormalization of this expression we can introduce the renormalization factors [8]

$$Z_e^{-1} = 1 + \frac{Q^2}{12\pi}\int_0^\infty \frac{\mathrm{d}s}{s}e^{-M^2 s}; \quad Z_g^{-1} = 1 - \frac{g^2}{12\pi}\int_0^\infty \frac{\mathrm{d}s}{s}e^{-M^2 s}, \tag{14}$$

which are generalizations of the definition Z_3 of Eq. (6). In this case the fields and charges are renormalized as [8]

$$E^2_{\rm reg} = Z_e^{-1}Z_g^{-1}E^2; \qquad H^2_{\rm reg} = Z_e^{-1}Z_g^{-1}H^2; \qquad e^2_{\rm reg} = Z_e Z_g e^2; \qquad g^2_{\rm reg} = Z_e^{-1}Z_g^{-1}g^2. \tag{15}$$

This relation (15) means that the vacuum of electrically charged particles shields the external electromagnetic field but the contribution from magnetically charged particles anti-shields it. This agrees with the results of [9] and [10].

Considering now the case of weak electromagnetic fields, the finite part of the Lagrangian $\Delta L'$, can, by analogy with (8), be written as

$$\begin{aligned}\Delta L' \;=\; & \frac{1}{360\pi^2 M^4}\big\{[(Q^2-g^2)^2 + 7Q^2g^2](\boldsymbol{H}^2 - \boldsymbol{E}^2)^2 + [16Q^2g^2 + 7(Q^2-g^2)^2](\boldsymbol{H}\boldsymbol{E})^2 \\ & +6Qg(Q^2-g^2)(\boldsymbol{H}\boldsymbol{E})(\boldsymbol{H}^2 - \boldsymbol{E}^2)\big\}. \end{aligned}\tag{16}$$

[3]We would like to stress that Eq. (10) is a postulate. For a discussion of the self-consistency of this approach, see, e.g. [1], [7], [8].

[4] This definition is consistent only if $\Box A_\mu = \Box B_\mu = 0$, i.e., for constant electromagnetic fields or for free electromagnetic waves.

The expressions (8) and (16) describe nonlinear corrections to the Maxwell equations which correspond to photon-photon interactions. The principal difference between the formula (16) and the standard Euler-Heisenberg effective Lagrangian consists in the appearance of P and T non-invariant terms proportional to $(\boldsymbol{HE})(\boldsymbol{H}^2 - \boldsymbol{E}^2)$. It should however be noted that this term is invariant under charge conjugation C, since then *both* Q and g would change sign.

Thus, the matrix element of the photon-photon interaction will contain terms which violate P and T.

If we consider separately the virtual creation of dyon pairs, then because of invariance of the model under a dual transformation (see, e.g. [1]) the physics is determined not by the values Q and g separately, but by the effective charge $\sqrt{Q^2 + g^2}$. In the same way the operations of P and T inversions are modified. However we will consider simultaneously the contributions from vacuum polarization by electron-positron and dyon pairs. In this case it is not possible to reformulate the theory in terms of just one effective charge by means of a dual transformation. Moreover the Dirac charge quantization condition connects just the electric charge of the electron and the magnetic charge of a dyon: $eg = n/2$ whereas the electric charge Q is not quantized.

It is widely believed, based both on experimental bounds and theoretical predictions [11] that the dyon mass would be large, $M \gg m$, where m is the electron mass. Thus, in the one-loop approximation the first non-linear correction to the QED Lagrangian from summing the contributions (8) and (16) can be written as

$$\Delta L' \approx \frac{e^4}{360\pi^2 m^4}\left[(\boldsymbol{H}^2 - \boldsymbol{E}^2)^2 + 7(\boldsymbol{HE})^2\right] + \frac{Qg(Q^2 - g^2)}{60\pi^2 M^4}(\boldsymbol{HE})(\boldsymbol{H}^2 - \boldsymbol{E}^2), \tag{17}$$

where the P and T invariant terms corresponding to vacuum polarization by dyons have been dropped because they are suppressed by factors M^{-4}. Thus, their contribution to the effective Lagrangian will be of the same order as that of the ordinary QED multiloop amplitudes which we neglect.

This expression (17) for the effective Lagrangian allows us to calculate the amplitude for photon splitting in an external non-spatially uniform magnetic field $\boldsymbol{H}$

$$\gamma(k) + \text{ external magnetic field } \boldsymbol{H} \rightarrow \gamma(k_1) + \gamma(k_2). \tag{18}$$

In order to determine this amplitude, we shall follow the approach by [12] (see also [4]). We may write the matrix element of the process in terms of functional derivatives of the effective Lagrangian:

$$\begin{aligned} M = {} & \delta E^i \delta E_1^j \delta E_2^k \frac{\delta^3 L}{\delta E^i \delta E^j \delta E^k}\bigg|_{\substack{\boldsymbol{E}=0\\ |\boldsymbol{H}|=H}} + \delta H^i \delta H_1^j \delta H_2^k \frac{\delta^3 L}{\delta H^i \delta H^j \delta H^k}\bigg|_{\substack{\boldsymbol{E}=0\\ |\boldsymbol{H}|=H}} \\ & + (\delta E^i \delta E_1^j \delta H_2^k + \delta E_1^i \delta E_2^j \delta H^k + \delta E_2^i \delta E^j \delta H_1^k)\frac{\delta^3 L}{\delta E^i \delta E^j \delta H^k}\bigg|_{\substack{\boldsymbol{E}=0\\ |\boldsymbol{H}|=H}} \\ & + (\delta H^i \delta H_1^j \delta E_2^k + \delta H_1^i \delta H_2^j \delta E^k + \delta H_2^i \delta H^j \delta E_1^k)\frac{\delta^3 L}{\delta H^i \delta H^j \delta E^k}\bigg|_{\substack{\boldsymbol{E}=0\\ |\boldsymbol{H}|=H}} \end{aligned} \tag{19}$$

where the subscripts 1 and 2 label the photons in the final state. So, the photons are just one-particle fluctuations over the considered vacuum field configuration, i.e., $\boldsymbol{E} = 0 + \delta \boldsymbol{E}$; $\boldsymbol{H} = \boldsymbol{H}_0 + \delta \boldsymbol{H}$, and the photon field strengths are

$$
\begin{aligned}
\delta H &= \sqrt{4\pi}\, \omega(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\varepsilon}}), & \delta E &= \sqrt{4\pi}\, \omega \hat{\boldsymbol{\varepsilon}}, \\
\delta H_1 &= \sqrt{4\pi}\, \omega_1(\hat{\boldsymbol{k}}_1 \times \hat{\boldsymbol{\varepsilon}}_1), & \delta E_1 &= \sqrt{4\pi}\, \omega_1 \hat{\boldsymbol{\varepsilon}}_1, \\
\delta H_2 &= \sqrt{4\pi}\, \omega_2(\hat{\boldsymbol{k}}_2 \times \hat{\boldsymbol{\varepsilon}}_2), & \delta E_2 &= \sqrt{4\pi}\, \omega_2 \hat{\boldsymbol{\varepsilon}}_2,
\end{aligned} \tag{20}
$$

Here we have not written out phase factors which are common to all terms and the unit vectors of the respective photon polarizations and momenta are $\hat{\boldsymbol{\varepsilon}}$, $\hat{\boldsymbol{\varepsilon}}_1$, $\hat{\boldsymbol{\varepsilon}}_2$ and $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{k}}_1$, $\hat{\boldsymbol{k}}_2$. For the case of small spatial variations of the external magnetic field, the change $\boldsymbol{p}$ of the total momentum would be rather small. The condition becomes [12]

$$
\omega = \omega_1 + \omega_2, \qquad \boldsymbol{p} + \boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2 \tag{21}
$$

Thus to the first order in the parameter $(|\boldsymbol{p}|/\omega)^{1/2}$ the angles between the photon momenta, given by $\cos\phi_1 = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}_1$, $\cos\phi_2 = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}_2$, $\cos\phi_{12} = \hat{\boldsymbol{k}}_1 \cdot \hat{\boldsymbol{k}}_2$, can be approximated as [12]

$$
\begin{aligned}
\phi_1 &\approx \left(\frac{\omega_2}{\omega_1}\right)^{1/2} \left(-2\frac{p}{\omega}\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{k}}\right)^{1/2}, \\
\phi_2 &\approx \left(\frac{\omega_1}{\omega_2}\right)^{1/2} \left(-2\frac{p}{\omega}\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{k}}\right)^{1/2}, \\
\phi_{12} &\approx \frac{\omega}{(\omega_1\omega_2)^{1/2}} \left(-2\frac{p}{\omega}\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{k}}\right)^{1/2} = \phi_1 + \phi_2,
\end{aligned} \tag{22}
$$

where use has been made of the fact that the final-state photons must lie in the plane defined by the magnetic field and the incident photon momentum.

The probability for photon splitting in an external field depends crucially on the polarizations of the photons [12], [4]. In the case of small-angle scattering, only the process $\gamma_{\parallel} \to \gamma_{\perp 1} + \gamma_{\perp 2}$ takes place. Here the indices $\parallel$ and $\perp$ correspond to polarization states where the direction of the magnetic field of the photon $\boldsymbol{H}_i = \sqrt{4\pi}\, \omega(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\varepsilon}})$ is either perpendicular to or parallel to the plane formed by the momentum of the initial photon $\boldsymbol{k}$ and the external magnetic field $\boldsymbol{H}$.

Thus after substitution of Eq. (17) into (19) and some lengthy but rather simple calculations we arrive at the expression for the matrix element for photon splitting in an external magnetic field, taking into account the extra term which corresponds to the dyon-loop contribution:

$$
M_{\gamma\to 2\gamma} = -\frac{4}{15\sqrt{\pi}} \left(\frac{e^4}{3m^4} \sin\theta \left[\frac{7}{4}(\phi_1^2 + \phi_2^2) - \phi_{12}^2 \right] - \frac{Qg(Q^2 - g^2)}{8M^4} \cos\theta\, \phi_{12}^2 \right) \omega\omega_1\omega_2 H \cos 2\beta \tag{23}
$$

where θ is the angle between the momentum $\boldsymbol{k}$ and the magnetic field $\boldsymbol{H}$, β the dihedral angle between the plane containing the above vectors and the one containing the momenta of the final-state photons, and the ϕ's given by Eq. (22).

As was noted above, the most important difference between Eq. (23) and the standard formula for the photon splitting matrix element [12] consists in the appearance of an additional P and T non-invariant term. Indeed, for the crossed process of two photons coalescing in an external field

$$\gamma(k_1) + \gamma(k_2) \rightarrow \gamma(k) + \text{ external magnetic field H} \tag{24}$$

the matrix element $M_{2\gamma\to\gamma}$ can be just obtained from Eq. (23) by reflection. Thus, the second term in (23) under this operation has to change its sign and we have

$$M_{2\gamma\to\gamma} = -\frac{4}{15\sqrt{\pi}}\left(\frac{e^4}{3m^4}\sin\theta\left[\frac{7}{4}(\phi_1^2+\phi_2^2)-\phi_{12}^2\right]+\frac{Qg(Q^2-g^2)}{8M^4}\cos\theta\,\phi_{12}^2\right)\omega\omega_1\omega_2 H\cos 2\beta \tag{25}$$

Thus, as a result of interference between two one-loop diagrams corresponding to loops with dyons and those with simply electrically charged particles there is an asymmetry between the processes of photon splitting and photon coalescence. The physical effect of this asymmetry will depend on the photon spectrum and the directions of the photon momenta with respect to the magnetic field. In particular, the asymmetry vanishes when these are perpendicular, i.e. for $\cos\theta = 0$. A simple measure is the factor of asymmetry [13], which for values of θ not too close to zero takes the form

$$\delta = \frac{|M_{2\gamma\to\gamma}|^2 - |M_{\gamma\to 2\gamma}|^2}{|M_{2\gamma\to\gamma}|^2 + |M_{\gamma\to 2\gamma}|^2} \simeq \frac{3Qg(Q^2-g^2)}{e^4}\,\frac{m^4}{M^4}\,\frac{\phi_{12}^2}{7(\phi_1^2+\phi_2^2)-4\phi_{12}^2}\cot\theta, \qquad \theta \geq \mathcal{O}\left(\frac{m}{M}\right)^4 \tag{26}$$

which, as expected, is linear in the product of the dyon charges, and proportional to the fourth power of the electron to dyon mass ratio. Possible consequences of this effect in the context of Early Universe evolution will be discussed elsewhere.

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